Stat 444
Advanced Long-term Actuarial Math

Lecture 1: Multiple-state Models

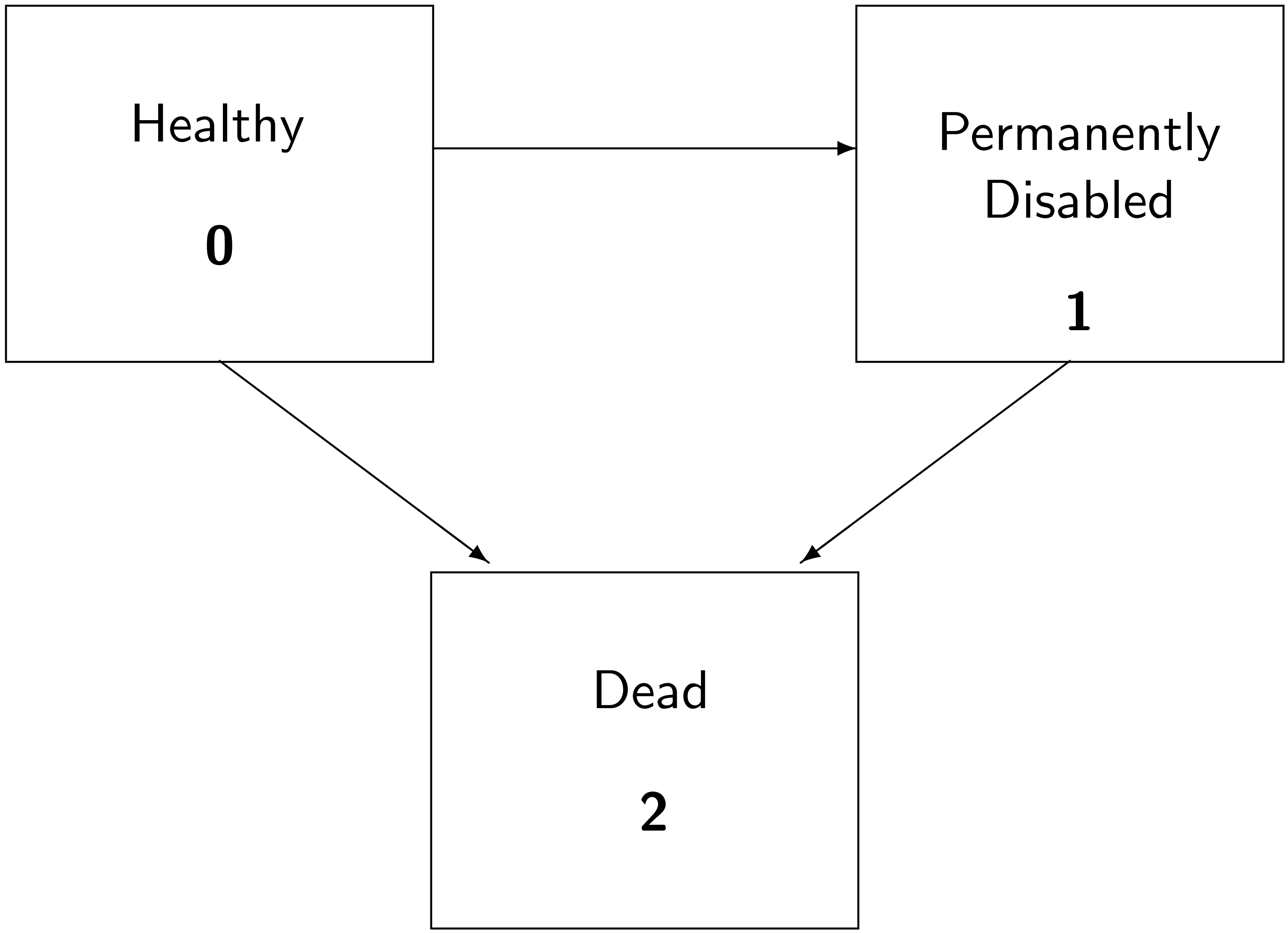
Multi-state Models

To this point, all of the products we've dealt with were based on the simple life status (alive / dead) of a single individual. Hence, the simple alive-dead model has been sufficient for our needs:

However, different situations and products can sometimes necessitate models that are more complex and/or general.

For example, if we were pricing a product that makes payments while the insured is permanently disabled, we would need a different type of model.

Multi-state Model Example — Permanent Disability



Some important multi-state model concepts

- Our multiple state models will all be assumed to possess the **Markov property**, whereby the probabilities of moving to the various states in the next time period depends only on the state we're currently in, not the previous states of the process.
  - This assumption will greatly simplify many of our calculations.

- In a **discrete time model**, transitions between states can only happen at discrete time intervals, e.g., the end of each year, start of each month, etc.
- In a **continuous time model**, transitions between states can happen at any time.

- If all of the probabilities of moving to the various states are constants whose values do not change over time, the process is a **homogeneous Markov process**; otherwise, it's known as **non-homogeneous**.

Discrete Time Models

For discrete time models, the probabilities of transitions between states can be described by a **transition probability matrix**. For a model with k states, this matrix has the following form:

$$\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1k} \\ p_{21} & p_{22} & \cdots & p_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ p_{k1} & p_{k2} & \cdots & p_{kk} \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} H & PD & D \end{array} \\ \begin{array}{c} H \\ PD \\ D \end{array} & \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0 & 0.9 & 0.1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

where p_{ij} is the probability of being in state j one period from now if we're currently in state i.

Note that each row of this matrix must sum to 1.

Discrete Time Model Example

A three state Markov process with states Healthy, Sick, and Dead is described by the monthly transition probability matrix below:

$H \to H \to S$ $\quad 0.7 \cdot 0.2 = 0.14$

$H \to S \to S$ $\quad 0.2 \cdot 0.6 = 0.12$

$$\begin{array}{c|ccc} & H & S & D \\ \hline H & 0.7 & 0.2 & 0.1 \\ S & 0.2 & 0.6 & 0.2 \\ D & 0 & 0 & 1 \end{array}$$

$S \to H \to S$ $\quad 0.2 \cdot 0.2 \quad 0.04$

$S \to S \to S$ $\quad 0.6 \cdot 0.6 \quad 0.36$

$S \to H \to H$ $\quad 0.2 \cdot 0.7 \quad 0.14$

$S \to S \to H$ $\quad 0.6 \cdot 0.2 \quad 0.12$

$\left.\begin{array}{l} S \to H \to D \\ S \to D \to D \\ S \to S \to D \end{array}\right\}$

(a) For a person who is currently healthy, calculate the probability they will be sick two months from now. [0.26]

(b) For a person who is currently sick, calculate the probability they will be alive two months from now. [0.66]

Now suppose we're pricing a product that makes a payment of \$100 at the start of each month that the insured is sick.

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}$$

- The product has a single premium due at issue and is only issued to people who are currently healthy.

$v = \frac{1}{1.01}$

- It's a short-term product that can only make payments at times 1 and 2 (in months).

$E(Y) = 100v(0.08) + 100v^2(0.14) + (100(v+v^2))(0.12)$

$E(Y^2) = 100^2v^2(0.08) + 100^2v^4(0.14) + (100(v+v^2))^2(0.12)$

Let Y denote the PV of benefit RV for this product. Assuming that $i^{(12)} = 0.12$:

(c) Calculate the net single premium for the product, i.e., $E[Y]$. [45.29]

0

$100v$ $\quad$ $H \to S \to H$, $S \to D$

(d) Calculate Var(Y). [4737.37]

$100v^2$ $\quad$ $H \to H \to S$

$100(v+v^2)$ $\quad$ $H \to S \to S$

(e) Calculate $P[Y > 99]$. [0.20]

(Continuous Time) Multi-state Models — Notation

The **force of transition** from state i to state j at age x is denoted μ_x^{ij} and is a generalization of the force of mortality in our simple alive/dead model.

${}_{t}p_{x}^{ij}$

- A continuous time model is specified by giving the various forces of transition between states. (This is the continuous time analog to the transition probability matrix.)

${}_{t}q_{x} = {}_{t}p_{x}^{01} = {}_{t}p_{x}^{AD}$

For an individual currently age x who is currently in state i, the probability they will be in state j at age $x + t$ is denoted ${}_tp_x^{ij}$.

The probability that this individual will stay continuously in state i from age x to age $x + t$ is denoted ${}_tp_x^{\overline{ii}}$.

Note that for any state i, we have ${}_tp_x^{\overline{ii}} \leq {}_tp_x^{ii}$. (Why?)

Model Specification Summary

The specification of our mortality / transition models depends on:

- Whether we're dealing with discrete vs continuous models, and
- If we're using the alive-dead model or a multi-state model

	Discrete	**Continuous**
Alive-dead	Mortality Table (q_x)	Force of Mortality (μ_x)
Multi-state	Probability Transition Matrix	Forces of Transition (μ_x^{ij})

(Continuous Time) Multi-state Models — Relationships

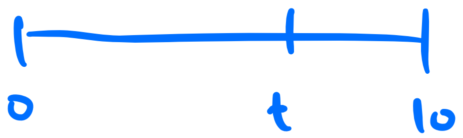
For any state i in a multi-state model with states $0 \ldots n$, we have the following relationships:

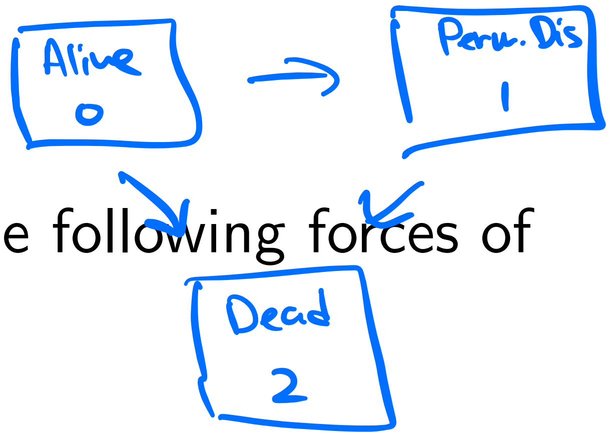
$$ {}_tp_x^{\overline{ii}} = \exp\left\{ -\int_0^t \sum_{j=0,\, j\neq i}^{n} \mu_{x+s}^{ij}\, ds \right\} $$

$$ {}_tp_x^{i0} + {}_tp_x^{i1} + \cdots + {}_tp_x^{in} = 1 $$

These are generalizations of more basic relationships in our alive-dead model.

For particular states i and j in a given model, we can find formulas for ${}_tp_x^{ij}$.





Use the permanent disability model with the following forces of transition:

$$\mu_x^{01} = 0.0279$$
$$\mu_x^{02} = 0.0229$$
$$\mu_x^{12} = 0.0229$$

$${}_tp_x^{\overline{ii}} = \exp\left\{-\int_0^t \sum_{j=0, j\neq i}^{n} \mu_{x+s}^{ij}\, ds\right\}$$

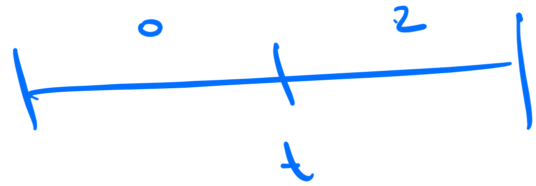
$$e^{-\int_0^{10} \mu_x^{01} + \mu_x^{02} dt}$$

(a) Calculate ${}_{10}p_{60}^{00}$. [0.60170]

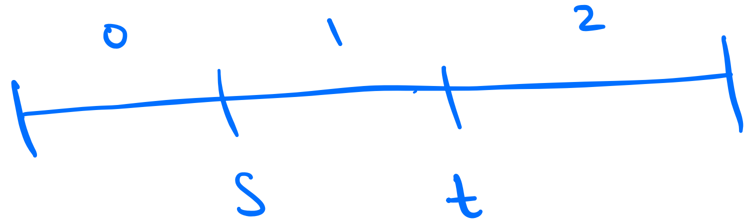
(b) Calculate ${}_{10}p_{60}^{01}$. [0.19363]

$$\int_0^{10} {}_tp_{60}^{00}\, \mu_{60+t}^{01}\, {}_{10-t}p_{60+t}^{11}\, dt = \int_0^{10} e^{-\int_0^t \mu_x^{01} + \mu_x^{02} ds} \cdot \mu_{60+t}^{01} \cdot e^{-\int_t^{10} \mu_x^{12} ds}\, dt$$

$${}_{10}p_x^{02}$$



$$\int_0^{10} {}_tp_x^{00}\,\mu_{x+t}^{02}\;{}_{10-t}p_{x+t}^{22}\,dt$$

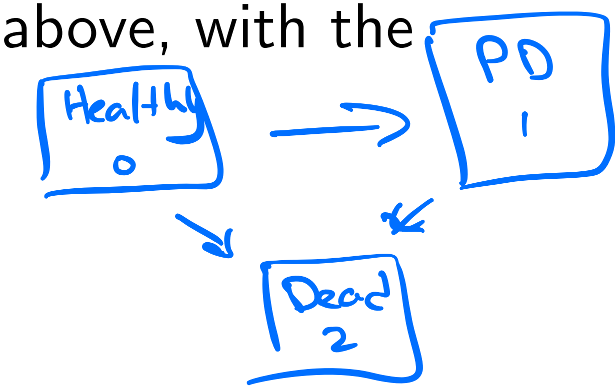


$$\int_0^{10}\int_0^t {}_sp_x^{00}\,\mu_{x+s}^{01}\;{}_{t-s}p_{x+s}^{11}\;\mu_{x+t}^{12}\;{}_{10-t}p_{x+t}^{22}\;ds\,dt$$

$$\int_0^{10}\int_0^t e^{-\int_0^s(\mu^{01}+\mu^{02})dr}\,\mu^{01}\,e^{-\int_s^t \mu^{12}dr}\,\mu^{12}\cdot 1\;ds\,dt$$

Continuous Time Multi-state Model Example 2

We're using the permanent disability model given above, with the following forces of transition:



$$\mu_x^{01} = 0.03$$
$$\mu_x^{02} = 0.02 + 0.002x$$
$$\mu_x^{12} = 0.02 + 0.004x$$

Assume Fred is 20 years old and is currently healthy.

(a) Calculate the probability that Fred will be healthy at age 25. [~~0.7985~~] 0.6219

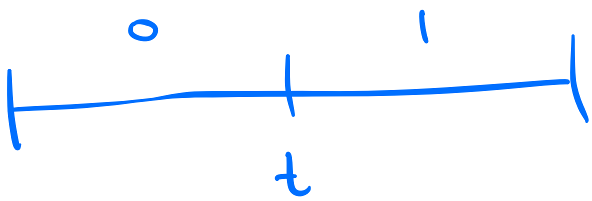
(b) Calculate the probability that Fred will be permanently disabled at age 23. [~~0.0823~~] ~0.0667

(c) Give an expression for the probability that Fred is dead at age 27.

$${}_5p_{20}^{\overline{00}} = e^{-\int_{20}^{25} \mu_x^{01} + \mu_x^{02}\,dx} = e^{-\int_{20}^{25} 0.05 + 0.002x\,dx}$$

$$= e^{-\int_0^5 \mu_{x+20}^{01} + \mu_{x+20}^{02}\,dx} = e^{-\int_0^5 0.05 + 0.002(x+20)dx}$$

$${}_3p_{20}^{01}$$



$$f(x) = {}_tp_x\, \mu_{x+t}^{01}(\ )$$

$$= \int_0^3 {}_tp_{20}^{00}\, \mu_{x+t}^{01}\ {}_{3-t}p_{20+t}^{11}\,dt$$

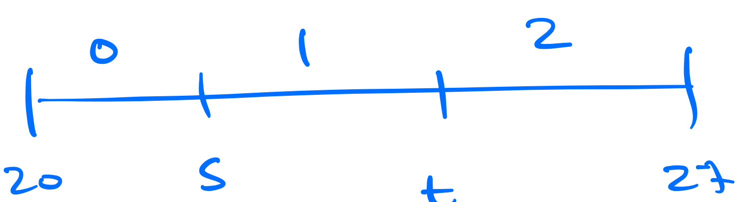
$$= \int_0^3 e^{-\int_0^t \mu_{20+s}^{01} + \mu_{20+s}^{02}\,ds}\ \mu_{20+t}^{01}\ e^{-\int_t^3 \mu_{20+s}^{12}\,ds}\,dt$$

$$= \int_0^3 e^{-\int_0^t 0.05 + 0.002(20+s)ds}\ (0.03) \cdot e^{-\int_t^3 0.02 + 0.004(20+s)\,ds}\,dt$$

$$e^{-\int_t^3 0.02 + 0.004(20+s)\,ds} = e^{-\int_0^{3-t} 0.02 + 0.004(20+t+s)ds}$$

$$\int_0^3 e^{-\int_0^t \mu^{01}_{20+s}+\mu^{02}_{20+s}\,ds}\,\mu^{01}_{20+t}\,e^{-\int_t^3 \mu^{12}_{20+s}\,ds}\,dt$$

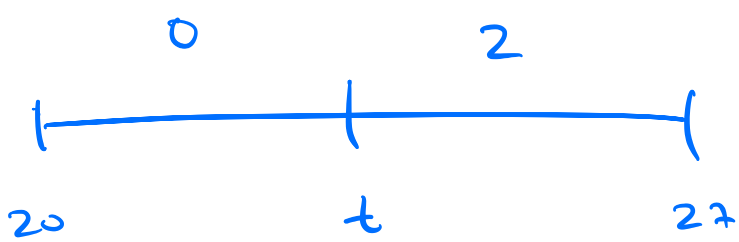
$$=\int_{20}^{23} e^{-\int_{20}^t \mu^{01}_s+\mu^{02}_s\,ds}\,\mu^{01}_t\,e^{-\int_t^{23}\mu^{12}_s\,ds}\,dt$$



(c)

$$\int_{20}^{27}\int_{20}^{t} e^{-\int_{20}^s \mu^{01}_r+\mu^{02}_r\,dr}\,\mu^{01}_s\,e^{-\int_s^t \mu^{12}_r\,dr}\,\mu^{12}_t\cdot 1\,ds\,dt$$

$$\int_0^7\int_0^t e^{-\int_0^s \mu^{01}_{20+r}+\mu^{02}_{20+r}\,dr}\,\mu^{01}_{20+s}\,e^{-\int_s^t\mu^{12}_{20+r}\,dr}\,\mu^{12}_{20+t}\,ds\,dt$$



$$\int_{20}^{27} e^{-\int_{20}^t \mu^{01}_r+\mu^{02}_r\,dr}\,\mu^{02}_t\cdot 1\,dt$$

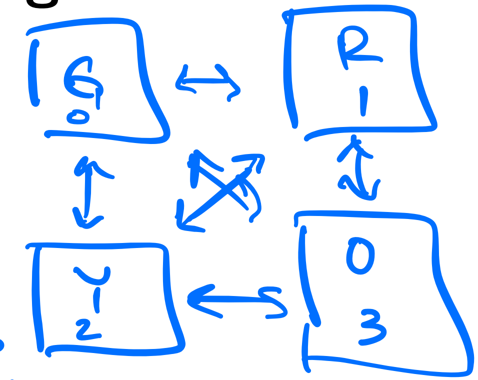
$$\int_0^7 e^{-\int_0^s \mu^{01}_{20+r}+\mu^{02}_{20+r}\,dr}\,\mu^{02}_{t+20}\,dt$$

For a multi-state model with states $0, 1, \ldots, n$, **Kolmogorov's Forward Equations** are given by

${}_tP_x^{03} = 0.2$

${}_{t+\epsilon}P_x^{03} = 0.2 + \alpha - \beta$

$$\frac{d}{dt}\,{}_tp_x^{ij} = \sum_{k=0,k\neq j}^{n} \left(\underbrace{{}_tp_x^{ik}\,\mu_{x+t}^{kj}}_{\text{people from other states who move in}} - \underbrace{{}_tp_x^{ij}\,\mu_{x+t}^{jk}}_{\text{people who moved out}} \right)$$



We can also apply Euler's (Forward) method to Kolmogorov's forward equations to get the following approximation for some small value of h:

$$ {}_{t+h}p_x^{ij} \approx {}_tp_x^{ij} + h \sum_{k=0,k\neq j}^{n} \left({}_tp_x^{ik}\,\mu_{x+t}^{kj} - {}_tp_x^{ij}\,\mu_{x+t}^{jk} \right)$$

Continuous Time Multi-state Model Example (continued)

Continuing the previous example,

(d) Using a step size of $h = 0.25$, calculate the probability that Fred is in each of the three possible states in 0.5 years.

Note the following:

- Many of the terms will be zero or one by definition. This will cause a number of terms to drop out.
- We have to do this calculation iteratively, first calculating the probability of being in states 0, 1, and 2 after one step (i.e., after 0.25 years).
  - Then we use these values to find the probability of being in state 2 after two steps (i.e., after 0.5 years), proceeding forward through time (see table).

Multi-state Model EPV Notation

There's no standard actuarial notation for life insurance or annuity EPVs in the multi-state model context, but our book introduces some useful notation. For example: $\bar{A}_{40} = \bar{A}_{40}^{01}$

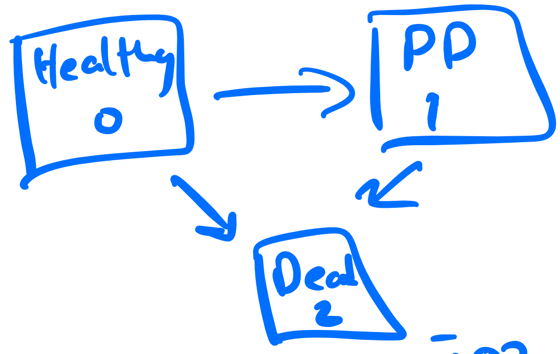
$\bar{A}_x^{ij}$ represents the EPV of a whole life insurance issued to someone currently age x and in state i, paying a benefit of 1 immediately upon the insured moving to state j.

$$\text{Generically, } \bar{A}_x^{ij} = \int_0^\infty e^{-\delta\, t} \sum_{k \neq j} {}_t p_x^{ik}\, \mu_{x+t}^{kj}\, dt$$

$\ddot{a}_{x:\overline{n}|}^{ij}$ represents the EPV of an n year term life annuity due issued to someone currently age x and in state i, paying an amount of 1 at the beginning of each year, provided that the annuitant is in state j at the time of payment. Payments stop after n years.

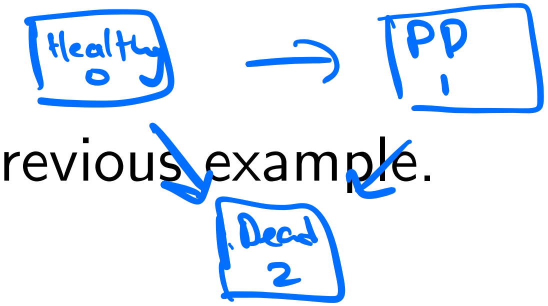
- Note here that i and j could be the same state.

$\ddot{a}_{40:\overline{10}|} = \ddot{a}_{40:\overline{10}|}^{00}$



$$EPV = \bar{A}^{02}_{x} + \frac{1}{2}\left(\bar{A}^{01}_{x} + \bar{A}^{12}_{x+t}\right)$$

Continuous Time Multi-state Model Example (continued)



Now we'll look at calculating EPVs for the previous example. Assuming a force of interest of δ:

(e) Give an expression for the EPV of a whole life insurance with \$1 death benefit issued to Fred. The insurance pays \$1 at the moment Fred enters the Dead state.

(f) Give an expression for the EPV of a whole life annuity due with payments of \$1 at the beginning of each year Fred is in the Healthy state.

(g) Give an expression for the annual net premium for Fred's life insurance policy, if premiums are only payable when Fred is in the Healthy state.

(e) $\bar{A}^{02}_{20} = \int_0^\infty e^{-\delta t}\left[{}_tP^{00}_{20}\, \mu^{02}_{20+t} + \int_0^t {}_sP^{00}_{20}\, \mu^{01}_{20+s}\; {}_{t-s}P^{11}_{20+s}\, \mu^{12}_{20+t}\, ds \right] dt$

(f) $\ddot{a}^{00}_{20} = \sum_{k=0}^{\infty} v^k\, {}_kP^{00}_{20} = \sum_{k=0}^{\infty} v^k e^{-\int_0^k \left(\mu^{01}_{20+s} + \mu^{02}_{20+s}\right) ds}$

(g) $P = \dfrac{\bar{A}^{02}_{20}}{\ddot{a}^{00}_{20}}$

$E(\text{In}) = P\ddot{a}^{00}_{20}$

$E(\text{Out}) = \bar{A}^{02}_{20}$

If you get 1/2 when disabled...

(e) $$\bar{A}^{02*}_{20} = \int_0^\infty e^{-\delta t} \, {}_tp^{00}_{20} \, \mu^{02}_{20+t} \, dt \; +$$

$$\int_0^\infty \int_0^t \tfrac{1}{2}\left(e^{-\delta t} + e^{-\delta s}\right) {}_sp^{00}_{20} \, \mu^{01}_{20+s} \; {}_{t-s}p^{11}_{20+s} \, \mu^{12}_{20+t} \, ds \; dt$$

Policy Values (Reserves) for Multi-state Models

In a multi-state model, the policy value has the same general propective definition as before, but its value will depend on which state we're currently in.

If a policy is in state j at time t, we denote its policy value by ${}_tV^{(j)}$ so that

$${}_tV^{(j)} = EPV_t(\text{future benefits}) \text{ - } EPV_t(\text{future premiums}),$$

where the EPV values are calculated based on being in state j at time t.

Policy Value Examples for Multi-state Models

${}_1V^{(D)} = 0$

For the sickness policy in the discrete time model example given above, calculate the net premium policy value at time 1 month, if at that time, the policyholder is:

$EPV(Ben) = v\, {}_1p_1^{HS}$

$EPV(Prem) = 0$

${}_1V^{(H)} = v(0.2) - 0$ } H

(a) Healthy

$EPV(Ben) = 1 + v(0.6)$

$EPV(Prem) = 0$

(b) Sick

${}_1V^{(S)} = 1 + 0.6v$

For the whole life insurance policy in the continuous time model example given above, give an expression for the net premium policy value at time 5, if at that time, the policyholder is:

(a) Healthy

(b) Permanently Disabled

${}_5V^{(0)} = \bar{A}_{25}^{02} - P\ddot{a}_{25}^{00}$

${}_5V^{(1)} = \bar{A}_{25}^{12} = \int_0^\infty e^{-\delta t}\, {}_tp_{25}^{11}\, \mu_{25+t}^{12}\, dt$

${}_5V^{(2)} = 0$

We can state a general form of Thiele's differential equation for our multi-state models: Death $S_t - {}_tV$

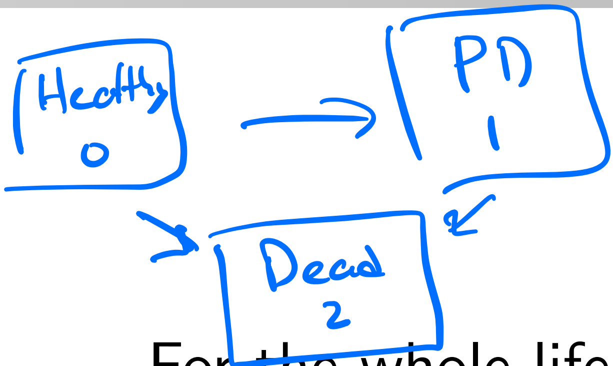
$$\frac{d}{dt}\,{}_tV^{(i)} = \delta_t\,{}_tV^{(i)} - B_t^{(i)} - \sum_{j=0,\,j\neq i}^{n} \mu_{x+t}^{ij}\left(S_t^{(ij)} + {}_tV^{(j)} - {}_tV^{(i)}\right),$$

where $B_t^{(i)}$ is the rate of benefit payable while the insured is in state i (including any associated expenses, and net of premiums), and $S_t^{(ij)}$ represents the lump sum payable (including associated expenses) upon transition from state i to state j.

- Note that many types of policies will not have both varieties of payment types.

We can apply Euler's method (typically the Backward version) to get discrete numerical values.

Thiele's DE for Multi-state Models — Example



$$\frac{d}{dt}\,{}_tV^{(i)} = \delta_t\,{}_tV^{(i)} - \underset{(S_t^i - P_t)}{B_t^{(i)}} - \sum_{j=0,\,j\neq i}^{n} \mu_{x+t}^{ij}\left(S_t^{(ij)} + {}_tV^{(j)} - {}_tV^{(i)}\right),$$

For the whole life insurance policy in the continuous time model, now assume that premiums are payable continuously while Fred is Healthy.

$$P = \frac{\bar{A}_{20}^{02}}{\bar{a}_{20}^{00}} = \frac{\int_0^\infty e^{-\delta t}\left({}_tp_{20}^{00}\,\mu_{20+t}^{02} + \int_0^t {}_sp_{20}^{00}\,\mu_{20+s}^{01}\,{}_{t-s}p_{20+s}^{11}\,\mu_{20+t}^{12}\,ds\right)dt}{\int_0^\infty e^{-\delta t}\,{}_tp_{20}^{00}\,dt}$$

(a) Find an expression for the net premium rate, P.

(b) Find expressions for $\frac{d}{dt}\,{}_5V^{(0)}$ and $\frac{d}{dt}\,{}_5V^{(1)}$.

$$\frac{d}{dt}\,{}_5V^{(0)} = \delta_5\,{}_5V^{(0)} + P - \sum_{j=1}^{2}\mu_{25}^{0j}\left(S_{25}^{0j} + {}_5V^{(j)} - {}_5V^{(0)}\right)$$

$$= \delta_5\,{}_5V^{(0)} + P - \mu_{25}^{01}\left({}_5V^{(1)} - {}_5V^{(0)}\right) - \mu_{25}^{02}\left(1 - {}_5V^{(0)}\right)$$

$$\frac{d}{dt}\,{}_5V^{(1)} = \delta_5\,{}_5V^{(1)} - \mu_{25}^{12}\left(1 - {}_5V^{(1)}\right)$$